THE TRANSFER FUNCTION APPROACH TO SENSITIVITY ANALYSIS BY DIGITAL COMPUTER*

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General Discussion of Sensitivity

Most modern system configurations consist of many interrelated components. These components may take the form of circuit parameters such as resistors, capacitors and amplifiers described by impedance, admittance and similar relationships or they may take the form of entire sub-systems consisting of a number of individual elements collected together as a functioning group and described by a transfer function. Whether we use impedance, admittance or transfer functions to express the operation of sub-system components, the concept of sensitivity remains unchanged, and this discussion applies equally to both. The importance of sensitivity as an analytic tool in understanding systems operation depends largely on its ability to show the relative importance of variations among components of a system. Sensitivity shows the effect on system performance of the variation of an element within the system. The term parameter in this discussion will be used to designate the element of interest whether it be a resistor value or some other quantile measuring sub-system characteristics.

Partial Derivatives

One method of measuring system sensitivity depends upon calculating the partial derivative of some system performance criterion with respect to a parameter variation. This is expressed mathematically as follows:

$$\mathcal{S}_{P}^{C} = \frac{\partial C}{\partial P} \qquad (1)$$

where C = a system performance criterion
P = a system parameter

This definition of sensitivity measures the incremental change in a system performance criterion C due to the change in performance parameter P with all other systems components held fixed. A simple example would be the variation of the

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*This paper is a synopsis of a larger paper which, for those interested in additional information, is available from the first author. Dr. S. Park Chan is associated with the Electrical Engineering Department of the University of Santa Clara.

gain of an amplifier circuit due to a change in the value of one resistor within the circuit.

Logarithmic Sensitivity

Developing the concept of sensitivity from one which measures the absolute change of a system performance criterion to one which measures the relative change of a performance criterion with respect to the relative change in a system parameter, we have the logarithmic sensitivity expressed as follows:

$$S^{C}_{P} = \frac{\ln C}{\ln P} = \delta^{C}_{P}\,\frac{P}{C} \qquad (2)$$

Logarithmic sensitivity is conveniently expressable as the percentage change in a performance criterion due to a percentage change in a parameter.

Root Sensitivity

The concept of root sensitivity developed in recent years as an outgrowth of the root locus method of system analysis describes the change in the location of system poles and zeros on the complex frequency plane as a result of the change of a system parameter. The subject is sufficiently broad that it would obscure the primary purpose of this discussion and will not be treated in further detail. For those interested in pursuing root sensitivity further, References [8] and [9] are recommended.

The Relationship of the Computer to Sensitivity Analysis

Because the computation of sensitivity is ordinarily a very tedious and error prone algebraic operation if done by hand, especially for large systems, the computer offers a great advantage; but the advantage is not obtained without difficulty. A systematic procedure must be developed to perform the required calculations in the computer and since digital computers do not conveniently differentiate the complex algebraic expressions representing system performance, this procedure must be structured around the more common numerical capabilities of computer operations. The prime objective of this discussion is to show how sensitivity and logarithmic sensitivity can be computed in a systematic way by means of computing certain transfer functions. The importance of this procedure hinges mainly on the fact that the problem of transfer function analysis has been well developed in computer technology. Many computer transfer function analysis programs have already been developed to a high degree of refinement and these programs can now be used directly in the study of sensitivities.

Mathematical Development

In order to show the mathematical development of several sensitivity relationships, the theory of signal flow graphs as developed by Mason[2,3], Coates[5], and others will be used. The signal flow graph is ideal in that it shows topologically the behavior of a system and the relationships among parameters within the system. For our purposes the performance criterion in signal flow graph terminology will be identical to the transmittance T selected for study within the system. With this in mind, we can start the development by revealing the formal mathematical description of the signal flow graph.

Any transmittance T of an arbitrary connected signal flow graph N can be expressed as a function of the gain of any edge P within the graph. This fact was stated by Mason[2] and Truxall[4] and a formal proof is given here.

Description of a Generalized Signal Flow Graph

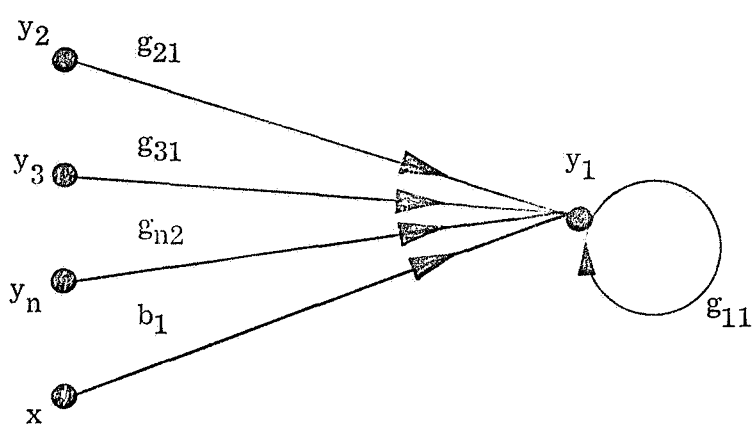

Figure 1

Generalized Signal Flow Graph Vertex

The value of any vertex y of a signal flow graph can be expressed as a linear combination of vertex value edge gain products as follows:

$$y_1 = g_{11}\, y_1 + g_{21}\, y_2 + g_{31}\, y_3 + \ldots + g_{n1}\, y_n - b_1 x \tag{3}$$

where the y's are dependent vertices and x is an independent vertex representing the input to the system or the source vertex.

If such a description is combined for all edges and vertices in a graph, we then have the set of all vertex values of a signal flow graph expressed as a combination of individual vertices in matrix form as follows:

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \cdots g_{1k} \cdots g_{1n} \\ g_{21} & g_{22} \cdots g_{2k} \cdots g_{2n} \\ \cdot & \\ \cdot & \\ \cdot & \\ g_{j1} & g_{j2} \cdots g_{jk} \cdots g_{jn} \\ \cdot & \\ \cdot & \\ \cdot & \\ g_{n1} & g_{n2} \cdots g_{nk} \cdots g_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} x \tag{4}$$

or $\quad Y = GY - BX$ and parameter $P = g_{jk}$

rearranging and simplifying we have:

$$\begin{bmatrix} (g_{11} - 1) & g_{12} & \cdots g_{1k} \cdots g_{1n} \\ g_{21} & (g_{22-1}) & \cdots g_{2k} \cdots g_{2n} \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ g_{j1} & g_{j2} & \cdots \; g_{jk} \cdots g_{jn} \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ g_{n1} & g_{n2} & \cdots \; g_{nk} \cdots (g_{nn-1}) \end{bmatrix} \begin{bmatrix} y_1/x \\ y_2/x \\ \cdot \\ \cdot \\ \cdot \\ y_k/x \\ \cdot \\ \cdot \\ \cdot \\ y_n/x \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} \tag{5}$$

or $\quad G' \; T = B$

where $T = (Y) \; 1/x$ and
$G' = (G - I)$

A solution for any transmittance $y_{m/x}$ can be found by solving this matrix equation as follows using Cramer's rule:

$$T = \frac{y_m}{x} = \frac{1}{\text{Det } G'} \text{ Det } \begin{array}{c} \begin{array}{ccccccc} \phantom{(g_{11}-1)} & \phantom{g_{22}-1} & & \overbrace{}^{\text{Col } m} & & \overbrace{\phantom{g_{1k}}}^{\text{Col } k} & \phantom{(g_{nn}-1)} \end{array} \\ \begin{bmatrix} (g_{11}-1) & g_{12} & \cdots & b_1 & \cdots & g_{1k} & \cdots & g_{1m} \\ g_{21} & g_{22-1} & \cdots & b_2 & \cdots & g_{2k} & \cdots & g_{2n} \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & & \cdot \\ g_{j1} & g_{j2} & \cdots & b_j & \cdots & g_{jk} & \cdots & g_{jn} \\ g_{n1} & g_{n2} & \cdots & b_n & \cdots & g_{nk} & \cdots & (g_{nn}-1) \end{bmatrix} \end{array} \tag{6}$$

This equation can be expanded around row j to yield the following result:

$$\frac{y_m}{x} = \frac{(-1)^{j+1} g''_{j1} M'_{j1} + (-1)^{j+2} g''_{j2} M'_{j2} + \ldots + (-1)^{j+k} g''_{jk} M'_{jk} + \ldots + (-1)^{j+n} g''_{jn} M'_{jn}}{(-1)^{j+1} g'_{j1} M_{j1} + (-1)^{j+2} g'_{j2} M_{j2} + \ldots + (-1)^{j+k} g'_{jk} M_{jk} + \ldots + (-1)^{j+n} g'_{jn} M_{jn}} \tag{7}$$

Where M'_{jp} = the (j,p) minor of the G' matrix with Col m replaced by B and $p = 1, 2, \ldots n$.

M_{jp} = the (j,p) the minor of G'

g'_{jp} = the (j,p) the element of G'

g''_{jp} = the (j,p) the element of the G' matrix with Column m replaced by B.

Since the expansion was made with respect to row j, all M' and M minors are devoid of g_{jk} and the expression may be rewritten as a linear function of g_{jk} as shown here.

$$\frac{y_m}{x} = \frac{a\, g_{jk} + b}{c\, g_{jk} + d} \tag{8}$$

where a, b, c, and d do not contain g_{jk}.

After inspection of Equation 7, we note that $g_{jk} = g'_{jk} = g''_{jk}$. This is so because all elements of G, G', and G'' are identical except those in Column m and those on the main diagonal representing self loops. For Column m, two cases exist, one in which g'_{jk} does not lie in Column m and one in which g'_{jk} does fall in Column m.

The first case guarantees that g''_{jk} is equal to g_{jk} if it does not fall on the main diagonal and this is true as shown below. The second case causes g''_{jk} to vanish by being replaced by b_j and thus the transfer function y_m/x can be considered a special case of Equation 8 with a, the coefficient of g_{jk} set equal to zero. Since it is not meaningful in a real system to consider a parameter which relates a network variable to itself, no self loop g_{jk} is possible and consequently Equation 8 is true generally. It follows directly that:

$$a = (-1)^{j+k} M'_{jk} \quad \text{where } K \neq m, j$$

$$b = (-1)^{j+1} g''_{j1} M'_{j1} + (-1)^{j+2} g''_{j2} M'_{j2} + \ldots + (-1)^{j+m} b_j M'_{jm} + \ldots + (-1)^{j+n} g''_{jn} M'_{jn}$$

$$\text{where } P \neq K$$

$$c = (-1)^{j+k} M_{jk} \qquad K \neq m, j$$

$$d = (-1)^{j+1} g'_{j1} M_{j1} + (-1)^{j+2} g'_{j2} M_{j2} + \ldots + (-1)^{j+n} g'_{jn} M_{jn}$$

$$\text{where } P \neq K$$

This is an important and useful result which allows us to consider a great conceptual simplification. We can now discuss a new signal flow graph representation of any arbitrary system which shows only the elements of interest in sensitivity analysis namely, the elements controlling the transmittance T with the parameter P explicitly represented as shown in figure 2. This figure graphically synthesizes the mathematical expression of equation (8) where a, b, -c, and 1/d are directed edges of the graph

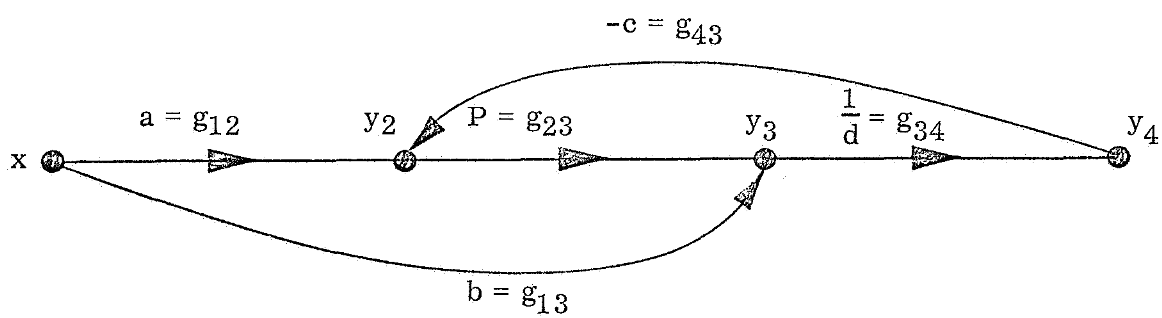

Figure 2

Simplified Signal Flow Graph Representing Any System and Showing Parameter P Explicitly

To prove this result we need only compute the transmittance through the system and arrange it in the form of equation (8):

$$T = \frac{y4}{x} = \frac{g_{12} P + g_{13}}{-g_{42} P + \frac{1}{g_{34}}} \tag{9}$$

identifying terms we have:

$$\begin{aligned} a &= g_{12} \\ b &= g_{13} \\ c &= -g_{42} \\ d &= \frac{1}{g_{34}} \end{aligned} \tag{10}$$

Partial Derivative

Using the graph of Figure 2, we can now proceed to develop sensitivity relationships which are significant in respect to any signal flow graph, indeed any system. By differentiating equation (9) we can derive an expression for partial derivatives which allows us the liberty of computing them by means of transfer functions alone.

Differentiating (9) with respect to P:

$$\frac{\partial T}{\partial P} = \frac{(g_{12} + g_{13}\, g_{34}\, g_{42})\, g_{34}}{(1 - g_{23}\, g_{34}\, g_{42})^2} \tag{11}$$

Expression 11 can be factored into two parts:

$\frac{\partial T}{\partial P} = T_1 T_2$ where T_1 and T_2 have the following significance:

$$T_1 = \frac{y_2}{x} = \frac{g_{12} + g_{13}\, g_{34}\, g_{42}}{1 - g_{23}\, g_{34}\, g_{42}} \tag{12}$$

= the transmittance of transfer function from the input vertex x of Figure 2 to the input vertex y_2 of the edge representing parameter P.

T_1 is directly interpretable as the transfer function from input to the parameter in question. For an example of T_1 consider any system voltage transfer function where the sensitivity with respect to a resistor in the system is sought. T_1 is the transconductance from system input to the resistor, i.e., the resistor current divided by the system input voltage which produces that current. Such a transfer function can be derived for any discrete parameter.

$$T_2 = \frac{y_4}{x'} = \frac{g_{34}}{1 - g_{23}\, g_{34}\, g_{42}} \tag{13}$$

= the transmittance or transfer function from the output vertex y_3 of the edge representing parameter P to the output vertex y_4 with the input x set to zero.

T_2 is similarly interpretable as the transfer function from the output of the parameter in question to the output of the system. In the example given above T_2 is the ratio of output voltage to the voltage across the resistor and again, such a transfer function can be found for any system parameter. Since the proof of this depends on the generalized flow graph representation of Fig. 2, where $P = g_{23}$ is an independent edge, it is necessary when one interprets this topological statement in terms of general system significance to preserve the multiplicative relationship $y_3 = P\, y_2$. This can be done for any parameter as shown in the example above where R, a resistance, is multiplied by a transconductance T_1 and a voltage transfer function T_2. Dimensional analysis indicates that a voltage transfer function should be produced which of course is true.

The justification for equation (13) can be demonstrated if we remove edges g_{12} and g_{13} from Figure 2, then add an artificial input vertex x' to form a new graph as shown in Figure 3. Equation (13) is the transmittance y_4/x' of this modified graph which is itself a topological statement of the method of computing the required factor.

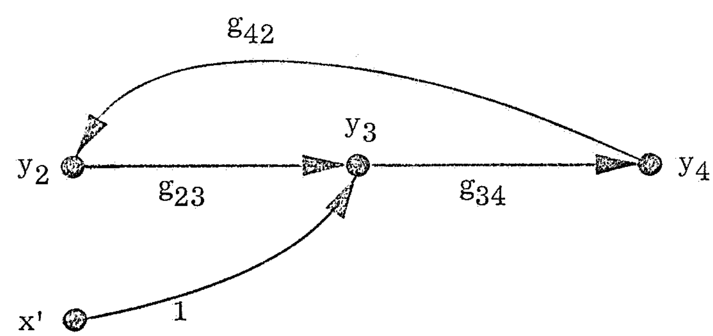

Figure 3

Sub-Graph Representing Part of the Partial Derivative Calculation by Means of Transmittance or Transfer Function Analysis

Logarithmic Sensitivity

Starting with the result obtained for absolute sensitivity of the generalized system as represented by the signal flow graph of Figure 2, we can now proceed to compute the logarithmic sensitivity. Repeating equation (11) we have:

$$\mathcal{S}^{T}_{P} = \frac{\partial T}{\partial P} = \frac{(g_{12} + g_{13}\, g_{34}\, g_{42})\,(g_{34})}{(1 - g_{23}\, g_{34}\, g_{42})^2} \tag{14}$$

The logarithmic sensitivity given in Equation 2 can be found by simply multiplying $\partial T/\partial P$ by P and dividing it by T as shown below:

$$S^{T}_{P} = \frac{d \ln T}{d \ln P} = \frac{\partial T}{\partial P}\,\frac{P}{T} \tag{15}$$

This form is altogether satisfactory for practical use in the computer if an algorithm for polynomial arithmetic is available. However, to show its relationship to the return difference sensitivity formula defined by Bode[1], we will proceed. Multiplying Equation 11 by P and dividing by T yields:

$$S^{T}_{P} = \frac{g_{12}\, g_{23} + g_{13}\, g_{23}\, g_{34}\, g_{42}}{(1 - g_{23}\, g_{34}\, g_{42})(g_{12}\, g_{23} + g_{13})} \tag{16}$$

After some algebraic manipulation, we have:

$$S^{T}_{P} = \frac{1}{1 - g_{23}\, g_{34}\, g_{42}} \left(1 - \frac{(g_{13}\, g_{34})\,(1 - g_{23}\, g_{34}\, g_{42})}{(g_{12}\, g_{23}\, g_{34} + g_{13}\, g_{34})}\right) \tag{17}$$

Each of the terms in the sensitivity equation in this form has a topological significance as follows:

$g_{23}\, g_{34}\, g_{42}$ = The transmittance from the input of parameter P (element g_{23}) through the network and back to the input node with the parameter P input terminal opened. The term $1 - g_{23}\, g_{34}\, g_{42}$ is the return difference defined by Bode. (18)

$$1 - g_{23}\, g_{34}\, g_{42} = \text{return difference} = RD \tag{19}$$

$g_{13}\, g_{34}$ = The transmittance of the network from input to output with P disconnected. (20)

$= T_{\bar{p}}$

$$\frac{1 - g_{23}\, g_{34}\, g_{42}}{g_{12}\, g_{23}\, g_{34} + g_{13}\, g_{34}} = \frac{1}{T} = \text{The reciprocal of the transmittance of the network.} \qquad (21)$$

Although this result is well known,* it is somewhat easier to evaluate the logarithmic sensitivity by the more direct method of multiplying the partial derivative by the parameter P and dividing by the transfer function T as outlined in equation (15).

Example of the Computation of Partial Derivative

In order to illustrate the significance of equations 11, 12, and 13 by means of an example, let us consider the simple electrical network of Figure 4 and the sensitivity of the network voltage transfer function V_o/V_{in} with respect to the capacitor C.

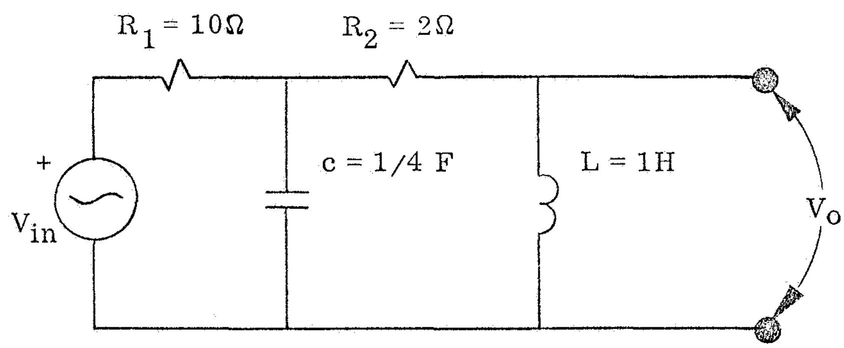

Figure 4

Simple Electrical Network

The first step in our process is to compute T_1. Since it is our objective to do this in a way that is suitable to a digital computer, we shall use one of many available transfer function analysis programs. Because the network is simple and computing time should be small and we can achieve the advantage of accuracy and numerical stability using the topological network analysis program of Calahan discussed in Reference [7]. Figure 5 shows the computer input and output for T_1 and Figure 6 shows the input and output for T_2.

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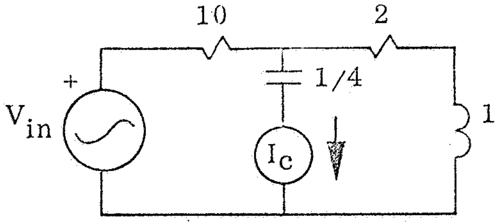
*See expression 4-22, page 58 of Reference [1]; see also expression 2-67, page 121 of Reference [4].



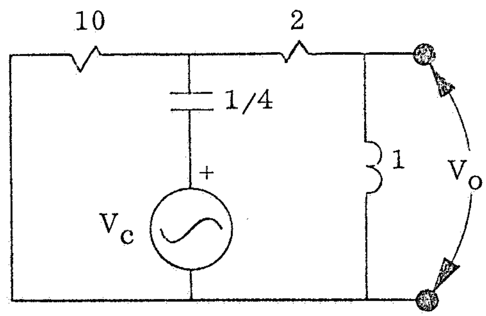
$$T_1 = \frac{I_c}{V_{in}} = \frac{0.1\ s\ (s + 2)}{(s + 1.2 \pm j\ 1.833)}$$

Input

Output

Figure 5

Computer Input and Output for T_1



$$T_1 = \frac{V_o}{V_c} = \frac{s^2}{(s + 1.2 \pm j\ 1.833)}$$

Input

Output

Figure 6

Computer Input and Output for T_2

and the partial derivative is found by taking the product of T_1 and T_2. Viz:

$$\delta\, \frac{T}{Z_c} = \frac{.1\ s^3\ (s + 2)}{(s + 1.2 \pm j\ 1.833)^2} \tag{22}$$

We can proceed to evaluate logarithmic sensitivity according to
follows:

$$\frac{T_1 \, T_2 \, Z_c}{T}$$

T in this case, is the voltage transfer function of Figu

$$\frac{V_o}{V_{i}}$$

Summary

The problem of computing sensitivities for any arbitrary system has been discussed. A mathematical development using the theory of signal flow graphs proves in a relatively simple way that sensitivity can be computed as the product of certain transfer functions derived from the system in question. This allows the use of general purpose transfer function computer analysis programs and reduces the problem of sensitivity to one that has already been solved. Further developments in extension of this theory to the general case of root sensitivity should be fruitful.

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